Semi-supervised Super-resolution for Gravitational Lenses with Estimated Degradation Model

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Abstract

High-resolution lensing images are essential in astrophysics for identifying and studying a range of physical phenomena, in particular the nature of dark matter. Deep learning approaches that learn super-resolution often require large amounts of training data. When the forward model (or degradation process) for super-resolution is known, model-based architectures such as loop unrolling have been shown to provide superior results and are more data-efficient than direct reconstruction methods. The classic loop unrolling requires knowing the forward model precisely, while a recent work addresses errors in forward model by iterative adaptation along reconstruction, achieving high reconstruction quality across various tasks but is designed for supervised learning only. High-resolution gravitational lensing images are expensive to obtain, but numerous low-resolution images are available. We propose to use an $\boldsymbol{A}$-adaptive loop unrolling architecture for high-resolution reconstruction, and incorporate the untrained adaptively estimated forward model network as part of the semi-supervised training loss. Experimental results demonstrate significant performance gains when leveraging the estimated forward model across different amounts of paired data.

1 Introduction

A gravitational lens occurs when a massive object, such as a galaxy cluster, lies between a distant source (like a galaxy) and an observer. Studying gravitational lenses provides insights into the distribution and nature of dark matter, the properties of distant galaxies, the validity of general relativity, and the structure and evolution of the cosmos [1, 2, 3, 4, 5, 6]. In particular, we aim to address the super-resolution (SR) problem in gravitational lensing. High-resolution images are essential for studying the intricate structures of distant galaxies, which enable detailed analysis and precise measurements that are not possible with low-resolution data – in particular, extracting properties of dark matter from substructure [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Machine learning has improved SR reconstruction but relies on large paired datasets. Due to limited high-resolution gravitational lens images from advanced equipment, supervised training no longer achieves optimal performance. However, with abundant low-resolution images, the goal is to enhance SR by leveraging these alongside the limited high-resolution data.

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In this project, we consider SR from the inverse problem perspective, which aims to recover the desired signal (high-resolution image $\boldsymbol{x}$) from noisy measurements (low-resolution image $\boldsymbol{y}$). The signals $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$ are related by the forward model $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, expressed as,

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\epsilon}, \tag{1}$$

where $\boldsymbol{\epsilon}$ is unknown observation noises. SR is an ill-posed problem because the reconstruction result is not unique, making it inherently challenging to solve.

A line of research aims to learn a direct inverse mapping from $\boldsymbol{y}$ to $\boldsymbol{x}$, such as variations of CNN [18, 19], generative adversarial networks [20] and diffusion model [21, 22]. When $\boldsymbol{A}$ is known, [23, 24, 25] incorporate $\boldsymbol{A}$ in self-supervised loss, i.e., $||\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{x}}||_2^2$ where $\hat{\boldsymbol{x}}$ is the estimated reconstruction. On the other hand, when $\boldsymbol{A}$ contains error or completely unknown, additional regularization is added to the mean-squared error (MSE) in the supervised learning loss $||\boldsymbol{x} - \hat{\boldsymbol{x}}||_2^2$. Hand-crafted regularizations, such as perceptual loss [26], total variation regularization [26], and contrastive loss [27], are typical choices independent of $\boldsymbol{A}$. While those hand-crafted regularizers can improve performance, they are task-specific and may not always yield the best possible results. In IPs, [28] learns the forward model together with a direct inverse mapping, and uses the approximated forward model f_ρ as part of the training loss, $||\boldsymbol{y} - f_\rho(\hat{\boldsymbol{x}})||_2^2$. In the experiments, we demonstrate that the proposed method adaptively estimates the forward model and outperforms this baseline.

Model-based architecture such as loop unrolling (LU) [29, 30, 31, 32] is considered to be more interpretable [31], more data efficient [31, 33], and achieves better reconstruction quality than other comparable black box data-driven approaches. LU addresses the suboptimal regularizer by iteratively refining the reconstruction, utilizing $\boldsymbol{A}$ at each iteration. Classical LU relies on accurate forward models. However, in gravitational lens SR problems, the forward model encodes the relationship between observations from different types of observatory equipment, making it challenging to formulate this exact mapping mathematically. Recent work [34] addresses forward model mismatch in LU using an untrained neural network that adaptively estimate the true forward model for each data instance during both training and inference, achieving performance comparable to LU with a known $\boldsymbol{A}$. In this work, we extend the adaptive LU to semi-supervised learning that incorporates the estimated forward model into the loss function. This paper is the first to address the semi-supervised SR problem using loop unrolling architecture with inaccurate forward models. We demonstrate that our method is more data-efficient than direct SR networks and its supervised loop unrolling counterpart, delivering superior reconstructions across varying levels of data availability.

2 Methodology

2.1 Supervised Loop Unrolling Methods

Loop unrolling (LU) method [29, 31, 35, 36, 37] is a class of model-based architectures in solving inverse problems in (1), which is inspired by classical optimization problems in solving for $\hat{\boldsymbol{x}}$,

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \frac{1}{2}||\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}||_2^2 + \gamma r(\boldsymbol{x}), \tag{2}$$

where r is an arbitrary regularization function and $\gamma > 0$ is the regularization coefficient. When the regularizer r is not differentiable, proximal gradient descent can be used to approximate $\hat{\boldsymbol{x}}$, for $k = 1, 2, 3, ...$ with step size η,

$$\boldsymbol{x}_{k+1} = prox(\boldsymbol{x}_k - \eta \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{x}_k - \boldsymbol{y})), \tag{3}$$

where the proximal operator replaced by a neural network. With a fixed number of iterations K, the output $\boldsymbol{x}_K$ is compared to the ground-truth $\boldsymbol{x}$, and the weights in $prox$ network are updated in supervised manner. LU is data efficient and achieves high-quality reconstructions but requires an accurate $\boldsymbol{A}$ to ensure correct and efficient learning.

2.2 A-adaptive loop unrolling method

A recent work [34] proposes a variation of the loop unrolling architecture that can adaptively update the forward model and the reconstruction alternatively. In this approach, an untrained neural network parameterized by θ, $netA_\theta : \boldsymbol{x} \mapsto \boldsymbol{y}$, is used to estimate the true forward model. The parameters θ are iteratively updated during the reconstruction of each data instance.

In SR problems, we assume $netA_\theta$ is used to approximate the true forward model, thus ideally, $\boldsymbol{y} = netA_\theta(\boldsymbol{x}) + \boldsymbol{\epsilon}$. Modified from the general $\boldsymbol{A}$-adaptive algorithms in [34], $\boldsymbol{A}$-adaptive LU for

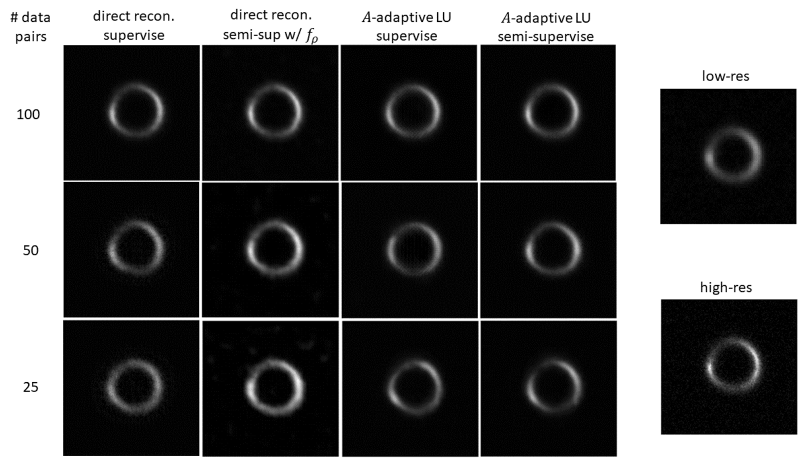

Figure 1: Visualization of super-resolution results using methods that can be trained without $\boldsymbol{A}$.

super-resolution aims to minimize the following objective function,

$$\min_{\boldsymbol{x},\theta} \frac{1}{2}\|\boldsymbol{y} - netA_\theta(\boldsymbol{x})\|_2^2 + \gamma r(\boldsymbol{x}), \tag{4}$$

where r is a regularization function and γ is a regularization coefficient that is set before training. Then, introduce an auxiliary variable $\boldsymbol{z}$, with initializations $\boldsymbol{x}_0, \boldsymbol{z}_0$ and θ_0, the optimal solutions can be obtained by updating each variable with step-size η for iterations $k = 1, 2, 3, ...$

$$\begin{aligned}
\boldsymbol{z}_{k+1} &= \arg\min_{z} \frac{1}{2}\|\boldsymbol{y} - netA_{\theta_k}(\boldsymbol{z})\|_2^2 + \lambda\|\boldsymbol{x}_k - \boldsymbol{z}\|_2^2, \\
\theta_{k+1} &= \arg\min_{\theta} \frac{1}{2}\|\boldsymbol{y} - netA_\theta(\boldsymbol{z}_{k+1})\|_2^2, \\
\boldsymbol{x}_{k+1} &= \text{prox}_{\frac{\gamma}{2\lambda}, r}(\boldsymbol{x}_k - \eta(\boldsymbol{x}_k - \boldsymbol{z}_{k+1})).
\end{aligned} \tag{5}$$

Notice that $netA_\theta$ is an untrained neural network, with θ being refined during $\boldsymbol{x}$ reconstruction. Backpropagation only learns the weights in the proximal network. The convergence of the algorithm in (5) is proved in [34]. Thus, when $\boldsymbol{z}_K$ and $\boldsymbol{x}_K$ are accurate estimates of $\boldsymbol{x}$, then $netA_\theta$ effectively emulates a reliable forward model, which allows us to use $netA_\theta$ as part of the loss for self-supervised learning.

2.3 A-adaptive loop unrolling method for semi-supervised learning

Previous works [38, 23] use the forward model as the training loss for self-supervised learning in the measurement domain. However, the forward model $\boldsymbol{A}$ for gravitational lensing SR problem is not known in practice. In this work, we aim to show that $\boldsymbol{A}$-adaptive method is not only powerful in reconstruction without knowing $\boldsymbol{A}$, but because $netA$ is an untrained neural network that allows for extensive estimation of the forward model for a specific data instance, eliminating the need to use many data pairs for training. This network can then be incorporated as part of the training loss for that particular data instance.

For the limited paired data, the network's training mirrors that of $\boldsymbol{A}$-adaptive LU. We propose the following semi-supervised learning strategy to incorporate the remaining unpaired data. As shown in [34], the A-adaptive LU not only ensures the convergence of the reconstruction $\boldsymbol{x}_k$, but also progressively refines the accuracy of the forward model approximation $netA$ over iterations. With the final estimate of the forward model $netA_{\theta_K}$, we define the semi-supervised training loss,

$$L = L_{\text{sup}} + L_{\text{self}} = \mathbb{E}_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{D}_{\text{sup}}} ||\boldsymbol{x} - \boldsymbol{x}_K(\boldsymbol{y})||_2^2 + \beta\, \mathbb{E}_{\boldsymbol{y}\in\mathcal{D}_{\text{self}}} ||\boldsymbol{y} - netA_{\theta_K}(\boldsymbol{x}_K(\boldsymbol{y}))||_2^2. \tag{6}$$

where $\boldsymbol{x}_K(\boldsymbol{y})$ denotes the output of $\boldsymbol{A}$-adaptive LU at final iteration K that is obtained from input $\boldsymbol{y}$, and β is a tunable training hyperparameter.

The supervised $\boldsymbol{A}$-adaptive LU minimizes the data consistency and the regularization function in (4). Yet, when with only limited paired data, this approach yields suboptimal reconstructions with

Table 1: Average testing PSNR and SSIM for super-resolution. The right-most column indicates the methods that can be trained without $\boldsymbol{A}$. Among the valid methods, the best performance in each criterion is in bold.

PSNR / SSIM	# data pairs	100	50	25	train w/o $\boldsymbol{A}$?
Direct recon.	supervised	29.24 / 0.586	27.46 / 0.544	26.53 / 0.527	✓
	semi-sup w/ $\boldsymbol{A}$	29.64 / 0.592	29.28 / 0.577	28.98 / 0.571	
	semi-sup w/ f_ρ	29.42 / 0.574	27.59 / 0.474	25.78 / 0.470	✓
LU	supervised	29.36 / 0.592	28.38 / 0.578	27.27 / 0.563	
	semi-sup w/ $\boldsymbol{A}$	29.97 / 0.594	29.96 / 0.594	29.29 / 0.575	
$\boldsymbol{A}$-adaptive LU	supervised	29.37 / 0.589	28.15 / 0.575	27.10 / 0.575	✓
	semi-sup w/ $\boldsymbol{A}$	**29.51 / 0.590**	**28.86 / 0.583**	**28.10 / 0.579**	✓

inaccuracies in both $||\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{x}}||_2^2$ and $r(\hat{\boldsymbol{x}})$. Incorporating unpaired data further reduces the error in $||\boldsymbol{y} - \boldsymbol{A}\hat{\boldsymbol{x}}||_2^2$, ensuring that the solution $\hat{\boldsymbol{x}}$ aligns with the forward model nullspace. Consequently, this facilitates more effective learning of the correct regularization function through supervised learning with ground-truth data. We further show that the semisupervised $\boldsymbol{A}$-adaptive LU is more effective than supervised methods even without knowing $\boldsymbol{A}$.

3 Experiments

We use the Model I dataset generated in [39] which consists of lensing images with different underlying dark matter model to validate the proposed approach. Specifically, this synthetic dataset consists of three classes of dark matter: no substructure, standard cold dark matter, and axion dark matter – see [39] for more details on the data set. Each category contains around 29,000 high-resolution and low-resolution image pairs. We use all low-resolution images for the self-supervised learning part and vary the amounts of high-resolution images for supervised training. We compare the proposed semi-supervised $\boldsymbol{A}$-adaptive LU with training loss in (6) to the following baseline methods, where the supervised learning methods only use a limited amount of paired data.

- Supervised Autoencoder that learns the direct reconstruction,
- Semi-supervised Autoencoder with known $\boldsymbol{A}$ in loss,
- Semi-supervised Autoencoder with jointly trained forward model network f_ρ [28] which is used in loss,
- Supervised LU with known $\boldsymbol{A}$ used in reconstruction,
- Semi-supervised LU with known $\boldsymbol{A}$ used in reconstruction and loss,
- Semi-supervised $\boldsymbol{A}$-adaptive LU with approximated forward model used in loss.

Table 1 illustrates the average testing peak signal-to-noise ratio (PSNR) and the structural similarity index (SSIM) values for different amounts of available data pairs. For all supervised learning scenarios, performance degrades as the number of data pairs decreases. Among the methods tested, supervised LU outperforms the others, showcasing its data efficiency through the use of an accurate forward model in both reconstruction and loss computation. Notably, even without knowing $\boldsymbol{A}$, $\boldsymbol{A}$-adaptive LU outperforms the direct reconstruction in supervised training, underscoring its ability to leverage adaptive estimation of the forward model. The right-most column in Table 1 highlights the methods trainable without $\boldsymbol{A}$. Among these three methods, the proposed $\boldsymbol{A}$-adaptive LU performs the best by incorporating the estimated forward model as part of the loss. Figure 1 visualizes the super-resolution results using the methods that can be trained without the forward model $\boldsymbol{A}$ (methods with checkmarks in Table 1). Notice that with fewer paired data, the reconstructions using direct inverse are fussier than the proposed method.

More results and discussions of using jointly trained forward models as loss are discussed in Appendix.

4 Conclusion

High-resolution gravitational lensing images are expensive to collect, but large amounts of low-resolution images are more readily available from survey programs like Euclid and LSST. Model-based architectures have been shown to be more data-efficient and achieve better quality than learning direct reconstructions, but they require accurate forward models in both the reconstruction process and the semi-supervised training loss. In this work, we proposed a semi-supervised learning method that leverages the $\boldsymbol{A}$-adaptive loop unrolling algorithm. Instead of learning a forward model directly from data in the traditional way, we use a dynamically adjusted forward model approximation network as part of the semi-supervised training loss. By approximating the forward model with an untrained neural network, our method avoids the need for extensive paired data to learn the forward mapping more accurately for each data instance, making it significantly more data-efficient than baseline methods. Lastly, we empirically demonstrate that the proposed method achieves significant performance gains.

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References

[1] L. V. E. Koopmans, A. Bolton, T. Treu, O. Czoske, M. W. Auger, M. Barnabè, S. Vegetti, R. Gavazzi, L. A. Moustakas, and S. Burles. The Structure and Dynamics of Massive Early-Type Galaxies: On Homology, Isothermality, and Isotropy Inside One Effective Radius. , 703(1):L51–L54, September 2009.

[2] Matteo Barnabè, Oliver Czoske, Léon V. E. Koopmans, Tommaso Treu, and Adam S. Bolton. Two-dimensional kinematics of SLACS lenses - III. Mass structure and dynamics of early-type lens galaxies beyond z 0.1. , 415(3):2215–2232, August 2011.

[3] Alessandro Sonnenfeld, Tommaso Treu, Raphaël Gavazzi, Sherry H. Suyu, Philip J. Marshall, Matthew W. Auger, and Carlo Nipoti. The SL2S Galaxy-scale Lens Sample. IV. The Dependence of the Total Mass Density Profile of Early-type Galaxies on Redshift, Stellar Mass, and Size. , 777(2):98, November 2013.

[4] Andrew B. Newman, Richard S. Ellis, and Tommaso Treu. Luminous and Dark Matter Profiles from Galaxies to Clusters: Bridging the Gap with Group-scale Lenses. , 814(1):26, November 2015.

[5] Rui Li, Yiping Shu, and Jiancheng Wang. Strong-lensing measurement of the total-mass-density profile out to three effective radii for z $\sim$ 0.5 early-type galaxies. , 480(1):431–438, October 2018.

[6] Amy Etherington, James W. Nightingale, Richard Massey, Andrew Robertson, XiaoYue Cao, Aristeidis Amvrosiadis, Shaun Cole, Carlos S. Frenk, Qiuhan He, David J. Lagattuta, Samuel Lange, and Ran Li. Beyond the bulge-halo conspiracy? Density profiles of early-type galaxies from extended-source strong lensing. , 521(4):6005–6018, June 2023.

[7] Tansu Daylan, Francis-Yan Cyr-Racine, Ana Diaz Rivero, Cora Dvorkin, and Douglas P. Finkbeiner. Probing the Small-scale Structure in Strongly Lensed Systems via Transdimensional Inference. , 854(2):141, February 2018.

[8] Johann Brehmer, Siddharth Mishra-Sharma, Joeri Hermans, Gilles Louppe, and Kyle Cranmer. Mining for Dark Matter Substructure: Inferring Subhalo Population Properties from Strong Lenses with Machine Learning. , 886(1):49, November 2019.

[9] Ana Diaz Rivero and Cora Dvorkin. Direct detection of dark matter substructure in strong lens images with convolutional neural networks. , 101(2):023515, January 2020.

[10] D. Bayer, L. V. E. Koopmans, J. P. McKean, S. Vegetti, T. Treu, C. D. Fassnacht, and K. Glazebrook. Probing sub-galactic mass structure with the power spectrum of surface-brightness anomalies in high-resolution observations of galaxy-galaxy strong gravitational lenses - I. Power-spectrum measurement and feasibility study. , 523(1):1326–1345, July 2023.

[11] Ran Li, Carlos S. Frenk, Shaun Cole, Liang Gao, Sownak Bose, and Wojciech A. Hellwing. Constraints on the identity of the dark matter from strong gravitational lenses. , 460(1):363–372, July 2016.

[12] Daniel Gilman, Simon Birrer, Tommaso Treu, Charles R. Keeton, and Anna Nierenberg. Probing the nature of dark matter by forward modelling flux ratios in strong gravitational lenses. , 481(1):819–834, November 2018.

[13] Stephon Alexander, Sergei Gleyzer, Evan McDonough, Michael W. Toomey, and Emanuele Usai. Deep Learning the Morphology of Dark Matter Substructure. , 893(1):15, April 2020.

[14] Atinç Çagan Şengül and Cora Dvorkin. Probing dark matter with strong gravitational lensing through an effective density slope. , 516(1):336–357, October 2022.

[15] S. Vegetti, S. Birrer, G. Despali, C. D. Fassnacht, D. Gilman, Y. Hezaveh, L. Perreault Levasseur, J. P. McKean, D. M. Powell, C. M. O'Riordan, and G. Vernardos. Strong gravitational lensing as a probe of dark matter. *arXiv e-prints*, page arXiv:2306.11781, June 2023.

[16] Noemi Anau Montel, Adam Coogan, Camila Correa, Konstantin Karchev, and Christoph Weniger. Estimating the warm dark matter mass from strong lensing images with truncated marginal neural ratio estimation. , 518(2):2746–2760, January 2023.

[17] Daniel Gilman, Yi-Ming Zhong, and Jo Bovy. Constraining resonant dark matter self-interactions with strong gravitational lenses. , 107(10):103008, May 2023.

[18] Chao Dong, Chen Change Loy, Kaiming He, and Xiaoou Tang. Image super-resolution using deep convolutional networks. *IEEE transactions on pattern analysis and machine intelligence*, 38(2):295–307, 2015.

[19] Chao Dong, Chen Change Loy, Kaiming He, and Xiaoou Tang. Learning a deep convolutional network for image super-resolution. In *Computer Vision–ECCV 2014: 13th European Conference, Zurich, Switzerland, September 6-12, 2014, Proceedings, Part IV 13*, pages 184–199. Springer, 2014.

[20] Christian Ledig, Lucas Theis, Ferenc Huszár, Jose Caballero, Andrew Cunningham, Alejandro Acosta, Andrew Aitken, Alykhan Tejani, Johannes Totz, Zehan Wang, et al. Photo-realistic single image super-resolution using a generative adversarial network. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 4681–4690, 2017.

[21] Chitwan Saharia, Jonathan Ho, William Chan, Tim Salimans, David J Fleet, and Mohammad Norouzi. Image super-resolution via iterative refinement. *IEEE transactions on pattern analysis and machine intelligence*, 45(4):4713–4726, 2022.

[22] Haoying Li, Yifan Yang, Meng Chang, Huajun Feng, Zhihai Xu, Qi Li, and Yueting Chen. Srdiff: Single image super-resolution with diffusion probabilistic models, 2021.

[23] Ortal Senouf, Sanketh Vedula, Tomer Weiss, Alex Bronstein, Oleg Michailovich, and Michael Zibulevsky. Self-supervised learning of inverse problem solvers in medical imaging. In *Domain Adaptation and Representation Transfer and Medical Image Learning with Less Labels and Imperfect Data: First MICCAI Workshop, DART 2019, and First International Workshop, MIL3ID 2019, Shenzhen, Held in Conjunction with MICCAI 2019, Shenzhen, China, October 13 and 17, 2019, Proceedings 1*, pages 111–119. Springer, 2019.

[24] Burhaneddin Yaman, Seyed Amir Hossein Hosseini, Steen Moeller, Jutta Ellermann, Kâmil Uğurbil, and Mehmet Akçakaya. Self-supervised learning of physics-guided reconstruction neural networks without fully sampled reference data. *Magnetic Resonance in Medicine*, 84(6):3172–3191, July 2020.

[25] Jingke Zhang, Qiong He, Congzhi Wang, Hongen Liao, and Jianwen Luo. A general framework for inverse problem solving using self-supervised deep learning: Validations in ultrasound and photoacoustic image reconstruction, 2021.

[26] Lerenhan Li, Yunlong Dong, Wenqi Ren, Jinshan Pan, Changxin Gao, Nong Sang, and Ming-Hsuan Yang. Semi-supervised image dehazing. *IEEE Transactions on Image Processing*, 29:2766–2779, 2019.

[27] Haiyan Wu, Yanyun Qu, Shaohui Lin, Jian Zhou, Ruizhi Qiao, Zhizhong Zhang, Yuan Xie, and Lizhuang Ma. Contrastive learning for compact single image dehazing. In *Proceedings of the IEEE/CVF conference on computer vision and pattern recognition*, pages 10551–10560, 2021.

[28] Motaz Alfarraj and Ghassan AlRegib. Semi-supervised learning for acoustic impedance inversion. In *SEG Technical Program Expanded Abstracts 2019*, pages 2298–2302. Society of Exploration Geophysicists, 2019.

[29] Karol Gregor and Yann LeCun. Learning fast approximations of sparse coding. In *Proceedings of the 27th international conference on international conference on machine learning*, pages 399–406, 2010.

[30] Zhaowen Wang, Ding Liu, Jianchao Yang, Wei Han, and Thomas Huang. Deep networks for image super-resolution with sparse prior. In *Proceedings of the IEEE international conference on computer vision*, pages 370–378, 2015.

[31] Vishal Monga, Yuelong Li, and Yonina C Eldar. Algorithm unrolling: Interpretable, efficient deep learning for signal and image processing. *IEEE Signal Processing Magazine*, 38(2):18–44, 2021.

[32] Peimeng Guan, Jihui Jin, Justin Romberg, and Mark A. Davenport. Loop unrolled shallow equilibrium regularizer (luser) – a memory-efficient inverse problem solver, 2022.

[33] Yuelong Li, Or Bar-Shira, Vishal Monga, and Yonina C Eldar. Deep algorithm unrolling for biomedical imaging. *arXiv preprint arXiv:2108.06637*, 2021.

[34] Peimeng Guan, Naveed Iqbal, Mark A. Davenport, and Mudassir Masood. Solving inverse problems with model mismatch using untrained neural networks within model-based architectures. *Transactions on Machine Learning Research*, 2024.

[35] Gregory Ongie, Ajil Jalal, Christopher A Metzler, Richard G Baraniuk, Alexandros G Dimakis, and Rebecca Willett. Deep learning techniques for inverse problems in imaging. *IEEE Journal on Selected Areas in Information Theory*, 1(1):39–56, 2020.

[36] Davis Gilton, Gregory Ongie, and Rebecca Willett. Deep equilibrium architectures for inverse problems in imaging. *IEEE Transactions on Computational Imaging*, 7:1123–1133, 2021.

[37] Seyed Amir Hossein Hosseini, Burhaneddin Yaman, Steen Moeller, Mingyi Hong, and Mehmet Akçakaya. Dense recurrent neural networks for accelerated mri: History-cognizant unrolling of optimization algorithms. *IEEE Journal of Selected Topics in Signal Processing*, 14(6):1280–1291, 2020.

[38] Gyeongha Hwang, Gihyeon Jeon, and Sunghwan Moon. Self-supervised learning for a nonlinear inverse problem with forward operator involving an unknown function arising in photoacoustic tomography. *arXiv preprint arXiv:2301.08693*, 2023.

[39] Stephon Alexander, Sergei Gleyzer, Hanna Parul, Pranath Reddy, Michael W. Toomey, Emanuele Usai, and Ryker Von Klar. Decoding dark matter substructure without supervision, 2021.

[40] Dmitry Ulyanov, Andrea Vedaldi, and Victor Lempitsky. Deep image prior. *International Journal of Computer Vision*, 128(7):1867–1888, March 2020.

A More Results

Figure 2 depicts the average testing PSNR for the four valid training without $\boldsymbol{A}$. The dashed lines in the bottom denote direct reconstruction methods, and the solid lines denote the $\boldsymbol{A}$-adaptive LU. With additional unpaired data, semisupervised $\boldsymbol{A}$-adaptive LU shows a significant improvement over its supervised counterpart, as indicated by the consistently higher average testing PSNR of super-resolution results. In contrast, the direct inverse network's performance does not show consistent enhancement, and the observed differences are relatively minor.

B Discussion of using a jointly trained forward model as loss

We would like to emphasize the difference between our proposed method and semi-supervised direct reconstruction with unknown $\boldsymbol{A}$. While both methods aim to approximate the forward model using a neural network, the semi-supervised direct reconstruction with unknown $\boldsymbol{A}$ method trains f_θ and the autoencoder jointly, but the proposed method uses an untrained network to estimate the forward model for each data instance. The proposed method outperforms the semi-supervised direct reconstruction with unknown $\boldsymbol{A}$, as shown in Table 1 and Figure 2. The neural network $netA_\theta$ leverages the complex structure itself to approximate the forward model, ensuring better alignment for a specific data instance. The idea of the untrained neural network is introduced in [40] to solve inverse problems. It allows $netA_\theta$ to serve as an accurate approximation in the semi-supervised training loss. By using an unknown neural network to approximate the forward model adaptively for each data instance, it can tailor the model more precisely to individual variations within the data. This personalized adjustment allows for better handling of complex patterns and anomalies that a more generalized model, like the semi-supervised direct reconstruction with unknown $\boldsymbol{A}$, may not capture as effectively.

In contrast, while the semi-supervised direct reconstruction with unknown $\boldsymbol{A}$ trains a forward model network and an autoencoder jointly, it may not offer the same level of individualized adaptation. The use of a learned forward model as a semisupervised training loss is less accurate than an adaptive approach, which could lead to suboptimal performance when dealing with diverse or unpaired data. This discrepancy highlights the advantage of our method.

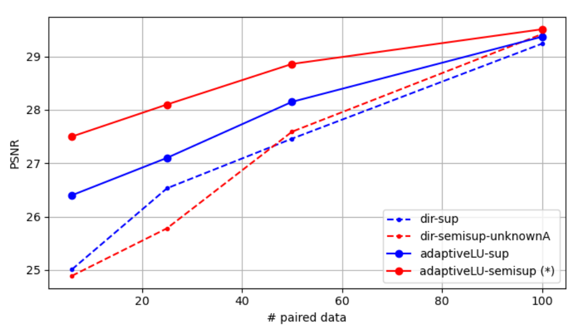

Figure 2: Average testing PSNR of super-resolution results using the four methods that can be trained without $\boldsymbol{A}$. The red curve is the proposed semisupervised adaptive LU, which performs the best with different amount of paired data.