Learning Conformal Field Theory with Symbolic Regression: Recovering the Symbolic Expressions for the Energy Spectrum

Haotian Cao
Department of Physics
University of Wisconsin-Madison
hcao39@wisc.edu

Garrett W. Merz
Data Science Institute
University of Wisconsin-Madison
garrett.merz@wisc.edu

Kyle Cranmer
Data Science Institute
University of Wisconsin-Madison
kyle.cranmer@wisc.edu

Gary Shiu
Department of Physics
University of Wisconsin-Madison
gshiu@wisc.edu

Abstract

In this paper, we employ Symbolic Regression (SR) to recover symbolic formulae for the primary conformal weights and central charges of two-dimensional rational conformal field theories (2D-RCFTs) given limited information about the energy spectra of these theories. We find symbolic expressions for these quantities in two well-understood classes of 2D-RCFT, minimal models and Wess-Zumino-Witten (WZW) $\hat{su}(2)_k$ models, and discuss future applications to theories about which less is known.

1 Introduction

Conformal Field Theory (CFT) is a type of Quantum Field Theory (QFT) which is invariant under conformal transformations. This enhanced symmetry places strong constraints on the theory such that some models can be exactly solvable in two-dimensions [3]. Two well-known types of such 2D Rational Conformal Field Theory (RCFT) with a finite number of primary operators are minimal models and Wess-Zumino-Witten (WZW) models. The minimal models are specifically generated by complex Lie algebras called Virasoro algebras [8, 10], while the WZW models are generated by Virasoro algebras with additional symmetry from affine Lie algebras such as $\hat{su}(2)_k$ [8]. In this context, one can determine physical quantities such as correlation functions, operator contents, and the partition functions explicitly [5].

Similar to quantum mechanics, one of the best ways to understand a conformal field theory is by studying its energy spectrum. In the context of 2D-RCFTs, understanding the energies can directly give us hints about the correlation functions [4, 8, 10] and the geometrical or topological properties [6, 12] of the theory. The energies in 2D-RCFTs usually take the form $E_n = h_n + N - \frac{c}{24}$ where h_n are called the primary conformal weights, N can be any positive integer, and c is the central charge that measures the degrees of freedom of the theory. Explicit formulas are known for both c and h_n [4, 8, 10]. However, both of these calculations require knowledge of the operator content of the theory, which may not be known in full for some classes of CFT.

The task of obtaining the explicit form of conformal weights given limited knowledge about the theory remains unexplored. Given recent advances in Symbolic Regression (SR), we apply pySR [7], a high-performance SR framework that returns human-interpretable expressions as outputs, to

38th Conference on Neural Information Processing Systems (NeurIPS 2024).

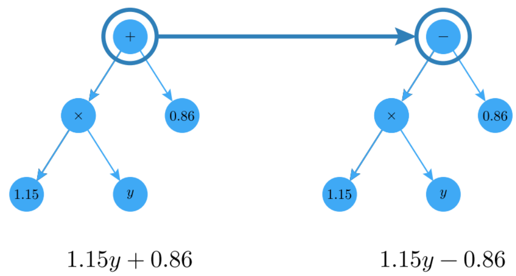

Figure 1: Mutation Operation on an expression tree [7].

recover the symbolic formula of c and h_n given the representation of the theory and its lowest few primaries. We will work on RCFTs as a first experiment to show that pySR has the potential to obtain more information about the spectrum of a theory based on the lowest energy states.

2 Related Work

The conformal bootstrap program, which dates back to the 1970s [9, 17, 18, 20, 21], is one of the most promising methods to extract exact or highly accurate information about the spectrum of anomalous scaling dimensions of conformal field theories, which are related to the lowest few h_n [3]. Recently, work using Reinforcement Learning (RL) has been performed to find numerical solutions of the 2D Ising model and 2D compactified boson in the context of the conformal bootstrap [14, 15].

The energy spectrum for non-rational CFTs with an infinite number of primary fields is usually determinable given the lowest few levels [3, 18]. Work using Principal Component Analysis (PCA) has shown that for a 3-manifold theory dual to a 2D logarithmic CFT, the lowest few energy gaps or consecutive energy ratios contribute the most to the entire energy spectrum [6, 12].

Several recent works have explored the application of symbolic regression techniques to "discrete" theoretical data where the inputs and outputs are exact and no noise is present. Examples of such work in this space includes the use of symbolic regression to find Green's functions for particular partial differential operators [11], to solve the Landau-Zener time evolution problem [2], and to characterize the topology of Calabi-Yau manifolds [1]. To our knowledge, our work is the first to apply symbolic regression to rational conformal field theories.

3 Method: Symbolic Regression

Symbolic regression is a regression analysis framework that attempts to approximate relations between inputs and outputs by searching the space of symbolic expressions instead of fitting individual parameters in a continuously parameterized model [7]. A wide variety of SR models and packages exist; recent approaches include Transformer-based models [13], reinforcement-learning based models [16], and evolutionary algorithm-based models [7, 19], among many others. In this work, we apply the open-source package pySR [7], which employs a tournament-style evolutionary algorithm to search for a set of equations that fit the data.

The model contains a set of parameters, operators, variables, and constraints which are combined into trees representing symbolic expressions. Through iterative processes of selection, crossover, and mutation, akin to natural evolution, the algorithm evolves the population over multiple generations and selects those that best fit the input data (as measured by minimizing the mean squared error). An example of the dynamics of crossover mutation on expression trees is illustrated in Figure 1. In constructing our trees, we employ the basic binary operators $+, -, \times$, and $\div$ for all models, as well as the unary operator "square" for the minimal models only.

In order to target our search towards gaining insight into the underlying structure of the theory [7], we wish to balance simplicity and accuracy, prioritizing expressions that are as concise as possible while still fitting the data well. In addition to mean squared error loss, we report "complexity" (number of nodes in the expression tree) and "score" (negative derivative of the log-loss with respect to complexity) for all selected expressions.

4 Data Generation

4.1 Setup

We are interested in the energy spectrum spanned by the primary conformal weights and central charge; these energies can be expressed as $E_n = h_n - \frac{c}{24}$. We observe that the spectrum contains both the primary conformal weights h_n and the descendant conformal weights $h_n + N$. Therefore, we apply pySR to the primary conformal weights only, as the rest of the spectrum can be generated by adding positive integers N.

For minimal models, the explicit expressions for c and $h_{r,s}$ are:

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{4pq}; h_{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{4pq} \tag{1}$$

where p and q are positive integers that are coprime and satisfy $\{p, q\} \geq 2$. The (r, s) pairs represent energy levels of the given theory and satisfy $1 \leq r \leq q - 1$ and $1 \leq s \leq p - 1$. Similarly, for WZW models, the explicit expressions for central charge and conformal weights are:

$$c_k = \frac{3k}{k+2}; h_l = \frac{l(l+2)}{4(k+2)} \tag{2}$$

where k is the positive integer that characterizes the WZW $\hat{su}(2)_k$ theory and l represents the energy levels and ranges from 0 to k.

4.2 Data

In total, we have four different sets of data(two types of CFT and two tasks). In all cases, the target variable is the last term in the PySR input expression (either c or h). In these experiments, we assume that the quantum numbers l and $[r, s]$ are known, which may not hold for an arbitrary CFT. In later sections, we will briefly mention the potential extension of our project to less understood CFTs.

Our first task is to obtain the expression for the central charge c. We randomly choose 100 conformal field theories with distinct pairs of p and q (in the case of minimal models) or distinct k (in the case of WZW models). Each distinct pair fixes a theory with a central charge c according to Equation 1 and Equation 2. Our data format is $[p, q, c_{p,q}]$ for minimal models and $[k, c_k]$ for WZW models.

Our second task is to obtain expressions for the primary conformal weights h_n. In analogy to similar recent work [12], we restrict our data to the lowest few energies and attempt to recover the formula for the primary conformal weights.

To construct our dataset, we include 100, 200, and 300 distinct theories with 10%, 30%, 50% lowest primary conformal weights from each theory. We monitor pySR's performance across the nine data regimes for each CFT family and discuss the results in the Appendix. Here, in analogy to similar recent work [12], we restrict our data to the lowest energies and attempt to recover the formula for the primary conformal weights. For all CFT families, we report our results on the 100 distinct CFTs with the smallest Coxeter numbers (k, in the case of WZW models and (p, q) in the case of minimal models), and of these we choose the lowest 10% of primary conformal weights (in the case of WZW models) or lowest 30% of primary conformal weights (in the case of minimal models). We report additional results using larger proportions of the dataset in the Appendix.

For the WZW models, we format the data as [k, l, h_l], where l indexes the individual states and h is the primary energy. For the minimal models, we format the input data as $[p, q, r, s, h_{r,s}]$, where the quantum number l is replaced by the tuple $[p, q]$. With no additional constraints, we recover the exact symbolic formula for the WZW models and a nearly-exact formula for the minimal models. If we impose an additional constraint encouraging the model to prioritize integers when building expression trees, we successfully recover the exact formula for minimal models as well.

Data	Symbolic Formula of c	MSE
WZW $\hat{su}(2)_k$ Model		
Exact Formula	$3k/(k+2)$	
pySR Input $[k]$	$(k+k+k)/(k+2.00)$	$\sim$ 1.01e-11
pySR Input $[k]$ Simplified	$3k/(k+2.00)$	Exact
Minimal Model		
Exact Formula	$1-6\frac{(p-q)^2}{pq}$	
pySR Input $[p,q]$	$(-1)\,(-1.00)+\frac{(p-q)^2}{q\frac{p}{-6.00}}$	$\sim$ 6.50e-13
pySR Input $[p,q]$ Simplified	$1.00-6.00\frac{(p-q)^2}{pq}$	Exact

Table 1: Symbolic expressions for central charge. Floating-point constants returned by PySR are here expressed to two decimal places. While the expressions recovered are exactly correct, the MSE is not exactly zero due to the precision of floating point operations.

5 Results

5.1 Recovering symbolic expressions for central charge

The exact symbolic formula for central charge can be discovered easily with minimal usage of operators and constraints. We included 400 n-iterations with a total population of 20, resulting a total of 8000 iterations in the model. Our results are summarized in Table 1. For WZW $\hat{su}(2)_k$ models, we applied the basic binary operators with unary operator "square." After around 2000 iterations, the exact formula $\frac{3k}{k+2}$is recovered. For minimal models, we adopt the basic binary operators along with the unary operator "square." We recover the exact formula $1-6\frac{(p-q)^2}{pq}$ after around 3900 iterations. For both classes of CFT, the runtime is approximately 16 minutes. All experiments in both this and the following section are performed on an ASUS TUF Dash F15 laptop with an Intel Core i7 CPU.

5.2 Recovering symbolic expressions for primary conformal weights

We summarize a set of symbolic expressions for both CFT models in Table 2. We select the model with 600 n-iterations and a total population of 20, corresponding to 12000 iterations. For both families of rational CFT studied, we present the complete symbolic formulae obtained from the data described in Section 4. We employ the "integer column" technique to extract symbolic formulae in the following manner: first, we limit the use of constant operators such as 0.249 by assigning them a high complexity penalty. Next, we concatenate positive integers $[1,...,10]$ to the data as additional input variables [7]. In this construction, PySR is strongly encouraged to choose its constants from the integers (or any combination of integers) in these columns. Hence, the resulting equation is more likely to be exact. We stress that this method is immediately applicable only in the regime where the target equation is known to be characterized by rational numbers (such as those governing rational CFTs), and may not extend as readily to the case of non-rational CFTs.

For WZW models, we first include the basic binary operators and unary operator "square" with no additional constraints. After simplification, the formula is the same as the exact one if we approximate 0.2502 to $\frac{1}{4}$ and 1.9942 to 2. When constants are restricted to the integers, we exactly recover the correct formula with very low MSE.

For the minimal models, the first result is obtained by the same application of constraints as for the WZW models. After simplification, the symbolic equation is close to the exact formula, except that the denominator of the first term is 2 instead of 4 (as shown in Table 2). As per results shown in the appendix, we find that the integer column technique is less effective for minimal models than for WZW models. In order to further reduce the complexity of the equation, we disallow the nested operator square(square(...)), and are at last able to successfully recover Equation 1. However, this

Data	Symbolic Formula of h	MSE
WZW $\hat{su}(2)_k$ Model		
Exact Formula	$\frac{l(l+2)}{4(k+2)}$	
pySR Input $[k, l]$	$\alpha_1 \frac{(\alpha_2+l)(\frac{l}{\alpha_3})}{(k+\alpha_2)}$	5.24e-10
pySR Input $[k, l]$, Simplified	$\frac{\alpha_1}{\alpha 3} \frac{l(l+\alpha_2)}{k+\alpha_2}$	
pySR Input $[k, l]$, with Integers	$\frac{l(l+2)}{4(k+2)}$	Exact
Minimal Model		
Exact Formula	$\frac{(pr-qs)^2-(p-q)^2}{4pq}$	
pySR Input:	$\frac{x1}{\frac{x0+x0}{(x2\frac{x0}{x1}-x3)^2}} - \frac{x0}{\beta_1(\frac{x0}{x0-x1})^2 x1}\beta_2$	3.489e-05
pySR Input: $[p, q, r, s]$, Simplified	$\frac{(pr-qs)^2}{2pq} - \frac{(p-q)^2}{\beta_1 pq}\beta_2$	
pySR Input: $[p, q, r, s]$, with Integers	$\frac{(-pr+qs)^2-(-p+q)^2}{4pq}$	Exact

Table 2: Symbolic Expressions for conformal weights. For ease of notation, floating-point constants returned by PySR are here expressed as variables: $\alpha_1 = 0.48822$, $\alpha_2 = 1.9952$, $\alpha_3 = 1.9514$ for the WZW models and $\beta_1 = 1.9896$, $\beta_2 = 0.49295$ for the minimal models.

clearly requires us to impose a strong external bias that may not be desirable for other theories- thus, we underscore that, when performing symbolic regression in a "discrete" regime where the loss curve may exhibit strong discontinuities (such as that of the rational CFTs), additional care must be undertaken to better understand the necessary complexity penalties required for specific types of mutation.

6 Conclusion and Outlook

We use the symbolic regression tool pySR to study minimal models and WZW models, two varieties of 2D rational conformal field theory. For a number of theories, we obtain symbolic formulae for the central charge using the integers that index the theories and for the primary conformal weights using these and the integers that index the lowest few energies. By adding a basis of integer columns together with a restriction on the nesting behavior of the "square" operator, pySR successfully picks the correct integers or combinations of integers to recover the exact symbolic formula for both models. We thus show that, under an appropriate set of constraints, pySR performs well with even limited information (lowest few conformal weights of the energy spectrum). When more conformal weights are added, we note that our performance may drop (see Appendix), suggesting that these constraints may be too strong outside the low-energy regime. More effort may therefore be needed in order to understand how best to adapt pySR to perform exact symbolic regression on rational function data.

An immediate extension of our work is to explore whether our method applies to other CFTs such as 3D Ising CFTs or $O(N)$ models without any quantum numbers or labels associated with the spectrum. In future work, we intend to explore these questions, as well as the applicability of other symbolic regression tools such as Transformers [22] to our task.

Acknowledgement and Disclosure of Funding

This work is partly supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics under Award Numbers DE-SC-0023719 and DE-SC-0017647. GM and KC are supported by the U.S. Department of Energy (DOE) under Award No. DE-FOA-0002705, KA/OR55/22 (AIHEP).

References

[1] Daattavya Aggarwal et al. "Machine learning Sasakian and G2 topology on contact Calabi-Yau 7-manifolds". In: *Physics Letters B* 850 (2024), p. 138517. ISSN: 0370-2693. DOI: https://doi.org/10.1016/j.physletb.2024.138517. URL: https://www.sciencedirect.com/science/article/pii/S0370269324000753.

[2] Sahel Ashhab. "Using machine learning to find exact analytic solutions to analytically posed physics problems". In: *Heliyon* 10.6 (Mar. 2024), e28124. ISSN: 2405-8440. DOI: 10.1016/j.heliyon.2024.e28124. URL: http://dx.doi.org/10.1016/j.heliyon.2024.e28124.

[3] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov. "Infinite conformal symmetry in two-dimensional quantum field theory". In: *Nuclear Physics B* 241.2 (1984), pp. 333–380. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(84)90052-X. URL: https://www.sciencedirect.com/science/article/pii/055032138490052X.

[4] Ralph Blumenhagen, Dieter Lüst, and Stefan Theisen. *Basic concepts of string theory*. Theoretical and Mathematical Physics. Heidelberg, Germany: Springer, 2013. ISBN: 978-3-642-29496-9. DOI: 10.1007/978-3-642-29497-6.

[5] John L. Cardy. "Operator content of two-dimensional conformally invariant theories". In: *Nuclear Physics B* 270 (1986), pp. 186–204. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(86)90552-3. URL: https://www.sciencedirect.com/science/article/pii/0550321386905523.

[6] Miranda C. N. Cheng et al. "3-Manifolds and VOA Characters". In: *Commun. Math. Phys.* 405.2 (2024), p. 44. DOI: 10.1007/s00220-023-04889-1. arXiv: 2201.04640 [hep-th].

[7] Miles Cranmer. *Interpretable Machine Learning for Science with PySR and SymbolicRegression.jl*. 2023. arXiv: 2305.01582 [astro-ph.IM]. URL: https://arxiv.org/abs/2305.01582.

[8] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997. ISBN: 978-0-387-94785-3, 978-1-4612-7475-9. DOI: 10.1007/978-1-4612-2256-9.

[9] S Ferrara, A.F Grillo, and R Gatto. "Tensor representations of conformal algebra and conformally covariant operator product expansion". In: *Annals of Physics* 76.1 (1973), pp. 161–188. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(73)90446-6. URL: https://www.sciencedirect.com/science/article/pii/0003491673904466.

[10] Paul Ginsparg. *Applied Conformal Field Theory*. 1988. arXiv: hep-th/9108028 [hep-th].

[11] Jianghang Gu et al. *Discovery of Green's function based on symbolic regression with physical hard constraints*. 2024. arXiv: 2408.00811 [physics.comp-ph]. URL: https://arxiv.org/abs/2408.00811.

[12] Sergei Gukov and Rak-Kyeong Seong. *Learning BPS Spectra and the Gap Conjecture*. 2024. arXiv: 2405.09993 [hep-th]. URL: https://arxiv.org/abs/2405.09993.

[13] Pierre-Alexandre Kamienny et al. *End-to-end symbolic regression with transformers*. 2022. arXiv: 2204.10532 [cs.LG]. URL: https://arxiv.org/abs/2204.10532.

[14] Gergely Kántor, Vasilis Niarchos, and Constantinos Papageorgakis. "Conformal bootstrap with reinforcement learning". In: *Phys. Rev. D* 105.2 (2022), p. 025018. DOI: 10.1103/PhysRevD.105.025018. arXiv: 2108.09330 [hep-th].

[15] Gergely Kántor, Vasilis Niarchos, and Constantinos Papageorgakis. "Solving Conformal Field Theories with Artificial Intelligence". In: *Phys. Rev. Lett.* 128.4 (2022), p. 041601. DOI: 10.1103/PhysRevLett.128.041601. arXiv: 2108.08859 [hep-th].

[16] Brenden K. Petersen et al. *Deep symbolic regression: Recovering mathematical expressions from data via risk-seeking policy gradients*. 2021. arXiv: 1912.04871 [cs.LG]. URL: https://arxiv.org/abs/1912.04871.

[17] David Poland, Slava Rychkov, and Alessandro Vichi. "The conformal bootstrap: Theory, numerical techniques, and applications". In: *Reviews of Modern Physics* 91.1 (Jan. 2019). ISSN: 1539-0756. DOI: 10.1103/revmodphys.91.015002. URL: http://dx.doi.org/10.1103/RevModPhys.91.015002.

[18] A. M. Polyakov. "Nonhamiltonian approach to conformal quantum field theory". In: *Zh. Eksp. Teor. Fiz.* 66 (1974), pp. 23–42.

[19] Michael Schmidt and Hod Lipson. "Distilling Free-Form Natural Laws from Experimental Data". In: *Science* 324.5923 (2009), pp. 81–85. DOI: 10.1126/science.1165893. eprint: https://www.science.org/doi/pdf/10.1126/science.1165893. URL: https://www.science.org/doi/abs/10.1126/science.1165893.

[20] Sheer El-Showk et al. "Solving the 3D Ising model with the conformal bootstrap". In: *Physical Review D* 86.2 (July 2012). ISSN: 1550-2368. DOI: 10.1103/physrevd.86.025022. URL: http://dx.doi.org/10.1103/PhysRevD.86.025022.

[21] David Simmons-Duffin. "The lightcone bootstrap and the spectrum of the 3d Ising CFT". In: *Journal of High Energy Physics* 2017.3 (Mar. 2017). ISSN: 1029-8479. DOI: 10.1007/jhep03(2017)086. URL: http://dx.doi.org/10.1007/JHEP03(2017)086.

[22] Ashish Vaswani et al. *Attention Is All You Need*. 2023. arXiv: 1706.03762 [cs.CL]. URL: https://arxiv.org/abs/1706.03762.

Percentage	100 Theories	200 Theories	300 Theories
No integer columns			
10%	9.449299e-12	8.151944e-12	1.159401e-09
30%	1.844283e-08	2.044487e-14	2.545054e-08
50%	1.279936e-11	5.608435e-14	8.549776e-12
Integer columns			
10%	0	0	1.119128e-16
30%	0	4.189906e-15	1.013663e-14
50%	0	2.367243e-14	0

Table 3: The **loss** of the output symbolic expressions for the WZW models.

A Additional Results

In this section, we present the complete result of conformal weight prediction on 100, 200, or 300 distinct theories, using the lowest 10%, 30%, or 50% of conformal weights. For each theory, we record the complexity, score, and loss in each of these nine data subsets in order to understand the performance of PySR across the different data regimes. We do not provide symbolic expressions recovered by PySR in these experiments.

A.1 The WZW $\hat{su}(2_k)$ Models

As shown in Table 3 and Table 4, across different numbers of distinct theories and percentages of the total energy spectrum within each theory, the loss for the non-integer datasets reduces to below 10^{-8}, which is small in magnitude compared to the individual conformal weights. However, as we add larger percentages of the spectrum to the input, the model generally shows increase in the complexity except for only the 300 theories with 50% spectrum. In comparison, when we adopt the integer column methods, the complexity decreases significantly, while the MSE loss drops quickly. For those datasets where the score is infinity and the loss is 0, the symbolic equations are exact.

A.2 The Minimal Models

Similar situations appear for the minimal models without the integer columns as shown in Table 5 and Table 6. The best data setup is found in 100 theories with 30% of the total spectrum with a loss of order 10^{-5}. This result is recorded in Section 5 as well. However, the dataset with integer columns doesn't work well as we would've expected. The loss of the discovered equations doesn't converge as much as in the WZW models. Moreover, the complexities of the discovered equations are still kept high as in non-integer columns case.

Hence, we restrict the nesting behavior of the "square" operator to search for the most straightforward representation while still fitting the data well. The result for 100 theories with 30% is shown in Section 5.

Percentage	100 Theories	200 Theories	300 Theories
No integer columns			
10%	10.470450 (14)	3.650134 (14)	0.468137 (19)
30%	0.304537 (23)	19.525989 (14)	1.892862e-01 (19)
50%	4.238845 (23)	20.04488 (14)	15.406462 (14)
Integer columns			
10%	inf (20)	inf (17)	24.646230 (11)
30%	inf (20)	21.91397 (11)	20.314145 (11)
50%	inf (12)	20.10332 (11)	inf (12)

Table 4: The **score** and **complexity** (in the parentheses) of the output symbolic expressions for the WZW models.

Percentage	100 Theories	200 Theories	300 Theories
No integer columns			
10%	0.001161	0.003779	6.590e-03
30%	3.489e-05	0.000613	0.001219
50%	0.000322	0.000601	0.001057
Integer columns			
10%	0.000962	0.004225	0.007488
30%	0.073412	0.374559	0.586004
50%	0.000264	0.001137	0.026394

Table 5: The **loss** of the output symbolic expressions for the minimal models.

Percentage	100 Theories	200 Theories	300 Theories
No integer columns			
10%	0.170389 (10)	0.980252 (9)	1.141e-01 (13)
30%	1.396000 (31)	0.267731 (34)	1.704845 (25)
50%	0.223155 (30)	3.411225e-01 (24)	3.090698e-01 (28)
Integer columns			
10%	0.207430 (12)	0.301836 (11)	0.439275 (11)
30%	0.084315 (23)	0.187518 (5)	0.586004 (13)
50%	0.17722 (31)	0.228162 (25)	0.026394 (32)

Table 6: The **score** and **complexity** of the output symbolic expressions for the minimal models.