Evolving graph codes with improved error thresholds for Pauli channels

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Abstract

Computing the quantum capacity of a quantum channel is challenging due to the phenomenon of superadditivity of coherent information. For a one-parameter family of quantum channels, such as the depolarizing channel, a closely related problem is that of determining the error threshold, defined as the largest noise level up to which reliable quantum error correction against i.i.d. noise remains possible. Graph states, which form a subclass of stabilizer states, have previously been shown to yield quantum codes with thresholds surpassing those of repetition codes for Pauli channels.

In this work, we propose the use of a genetic algorithm to discover new families of graph states that have a higher error threshold than repetition codes for Pauli channels. Our findings include, for the depolarizing channel, a graph state exhibiting an error threshold higher than the 5-qubit repetition code with only 10 channel qubits and one reference qubit. Additionally, we present novel families of graph-state codes for the BB84 channel and the 2-Pauli channel achieving higher thresholds than repetition codes. These results expand the set of graph-structured quantum codes that have positive coherent information beyond the hashing bound for special classes of Pauli channels.

1 Introduction

In his seminal 1948 work [1], Claude Shannon laid the foundations of information theory by answering a fundamental question: at what maximum rate can information be sent reliably over a noisy communication channel? The answer is given by the *channel capacity*, C, which characterizes a noisy channel's transmission capability. For a classical channel, the capacity is determined by maximizing the *mutual information* $I(X;Y)$ between the channel input X and output Y over all possible input distributions. This characterization is powerful because of its *additivity*: the capacity of using two independent channels is simply the sum of their individual capacities. Consequently, the capacity of using a single channel n times is just n times its single-use capacity. This property simplifies the computation of capacity to a single-letter optimization problem.

One analogous task in quantum information theory is to determine the *quantum capacity*, $Q(\mathcal{N})$, of a quantum channel $\mathcal{N}$, which quantifies the maximum rate of reliable quantum information transmission through the channel. Naively, one might expect to find $Q(\mathcal{N})$ by maximizing a quantum analogue of mutual information, known as the *coherent information*, I_c, over all possible single-qubit input states. However, this approach fails due to a striking quantum phenomenon known as *superadditivity* of coherent information.

Unlike the mutual information for classical channels, the coherent information is not additive. The rates achievable by using a channel n times in parallel can be strictly greater than n times the rates achievable with a single channel copy: $I_c(\mathcal{N}^{\otimes n}) > nI_c(\mathcal{N})$. This implies that encoding

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information across entangled states that span multiple channel copies can achieve communication rates unachievable when using a single copy of the channel. This necessitates a regularization over all possible block lengths in the definition of the quantum capacity of a quantum channel:

$$Q(\mathcal{N}) = \sup_n \frac{1}{n} I_c(\mathcal{N}^{\otimes n})$$

This unbounded optimization makes direct computation of $Q(\mathcal{N})$ intractable for all but a few families of channels. An even more extreme manifestation of non-classical behavior is *superactivation* of the quantum capacity, where two distinct channels, each with zero quantum capacity ($Q(\mathcal{N}_1) = 0$ and $Q(\mathcal{N}_2) = 0$), can yield a positive capacity when used together, $Q(\mathcal{N}_1 \otimes \mathcal{N}_2) > 0$.

Given these challenges, a primary goal in quantum information theory is to find good lower bounds on $Q(\mathcal{N})$ by designing effective quantum error-correcting codes. A positive lower bound for the quantum capacity of a channel proves its utility for quantum communication. One promising family of codes is built from *graph states*, a class of multipartite quantum states whose structure is described by a graph. A family of two-level tree graph states has previously been shown in [5] to yield thresholds for Pauli channels better than the single-use coherent information threshold, and even those of repetition codes. In this work, we propose the use of a genetic algorithm to discover novel graph state codes that provide improved error thresholds beyond those of repetition codes for several important Pauli channels.

2 Background

The *coherent information* of a channel $\mathcal{N} : A \to B$ is defined by maximizing over purifications $|\psi\rangle_{RA}$ of possible input states $\psi_A = \mathrm{tr}_R |\psi\rangle_{RA} \langle\psi|_{RA}$ [11,12]:

$$I_c(\mathcal{N}) = \max_{|\psi\rangle_{RA}} [S(\mathcal{N}(\psi_A)) - S(\mathcal{N}(\psi_{RA}))], \tag{1}$$

where $\mathcal{N}(\psi_{RA})$ denotes the state $(\mathrm{id}_R \otimes \mathcal{N})(|\psi\rangle_{RA} \langle\psi|_{RA})$ representing the output after the channel acts only on system A, and $S(\rho_A)$ is the von Neumann entropy of a state ρ_A. We then have the following expression for the quantum capacity of $\mathcal{N}$ [2-4]:

$$Q(\mathcal{N}) = \sup_{n \in \mathbb{N}} \frac{1}{n} I_c(\mathcal{N}^{\otimes n}). \tag{2}$$

A notorious example of a channel whose quantum capacity is unknown is the depolarizing channel, which is defined by $D_q(\rho) = (1-q)\rho + q\,\mathrm{tr}(\rho)\frac{1}{2}\mathbb{1}$ for $q \in [0, \frac{4}{3}]$. The quantum capacity of this channel in the range $q \in (0, \frac{1}{3})$ is unknown. The depolarizing channel is an example of a qubit Pauli channel, which is a class of channels defined by

$$N(\rho) = (1 - p_x - p_y - p_z)\rho + p_x X\rho X + p_y Y\rho Y + p_z Z\rho Z, \tag{3}$$

where X, Y and Z are Pauli matrices. When $p_x = p_y = p_z = \frac{p}{3}$, we obtain the depolarizing channel with $q = \frac{4p}{3}$. Two other one-parameter families of Pauli channels of interest are the BB84 (which correspond to independent bit and phase flips) and 2-Pauli channels (which correspond to only X and Z errors occurring with equal probability), which have $(p_x, p_y, p_z) = (p - p^2, p^2, p - p^2)$ and $(p_x, p_y, p_z) = (\frac{p}{2}, 0, \frac{p}{2})$ respectively. The quantum capacity of both these channels is also unknown. For one-parameter families of channels, a related open problem is that of determining the *threshold* of the channel, which is defined as the parameter value at which the quantum capacity vanishes. One interpretation of the threshold of a channel is as the largest noise level up to which reliable quantum error correction against i.i.d. noise remains possible.

The coherent information for one copy of a Pauli channel with Pauli error probabilities p_x, p_y, p_z is given by $I_c(\mathcal{N}) = 1 - H(\mathbf{p})$, where $H(\mathbf{p})$ is the Shannon entropy of the probability distribution $(1 - p_x - p_y - p_z, p_x, p_y, p_z)$. This gives a quantum capacity threshold called the *hashing bound* [8]. Topological quantum error-correcting codes achieving this hashing bound for Pauli channels are known, and there is even numerical evidence that the $XZZX$ surface code has a threshold exceeding the hashing bound for X-biased or Z-biased Pauli noise [9].

The best known thresholds for the depolarizing channel use concatenated codes [6,10]. Graph states, which form a subclass of stabilizer states, have also previously been shown to yield quantum codes with thresholds surpassing those of repetition codes for many Pauli channels [5,7]. A notable property of many examples of input states achieving superadditivity is that they have some form of symmetry.

2.1 Coherent information of graph states

For the objective function of our optimization, we use the coherent information of a graph state $|\Gamma_{AR}\rangle$ which is input to multiple copies of a Pauli channel. Graph states are specific multipartite entangled states defined by a graph Γ on $k = k_A + k_R$ vertices, partitioned between system A and reference R. Their structure simplifies analysis: a graph state translates any Pauli error into a Z-type error. When $|\Gamma_{AR}\rangle$ passes through $\mathcal{N}_{\mathbf{p}}^{\otimes k_A}$ (acting only on A), the resulting state σ_{BR} and its marginal $\sigma_B = \operatorname{tr}_R \sigma_{BR}$ remain diagonal in bases naturally associated with the graph, with eigenvalue distributions λ_B and λ_{BR} respectively. This reduces the calculation to computing Shannon entropies of λ_B and λ_{BR}, which we do following the algorithm described in [5, Algorithm 1].

3 Genetic algorithm

The space of all possible graphs for a fixed number of vertices is large, making exhaustive search impossible beyond a small number of qubits. We therefore employ a genetic algorithm, which is a bio-inspired heuristic search, to probe this space. The algorithm evolves a population of candidate graphs over many generations. At each step, graphs with higher coherent information (better "fitness") are more likely to be selected as parents, creating offspring for the next generation via crossover and mutation operators that preserve graph connectivity. For parent selection, we use tournament selection, where a small subset of graphs is randomly chosen, and the fittest from that group becomes a parent. Offspring are then created via uniform crossover, where each edge in the new graph is inherited from one of the two parents with equal probability. This process drives the population toward highly optimized solutions.

Algorithm 1 Genetic Algorithm

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Initialize population P_0 with N random, connected graphs.
Evaluate fitness f(G) = I_c(G) for all G ∈ P_0.
Initialize Hall of Fame H with the best graph from P_0.
for t = 1 to Generations do
    Select parents P' from P_{t-1} via tournament selection.
    Create offspring P_offspring via crossover on pairs from P'.
    With probability p_m, apply a mutation to each graph in P_offspring.
    Re-evaluate fitness and update Hall of Fame H.
    Set new population P_t ← P_offspring.
end for
return the best graph G_best ∈ H.
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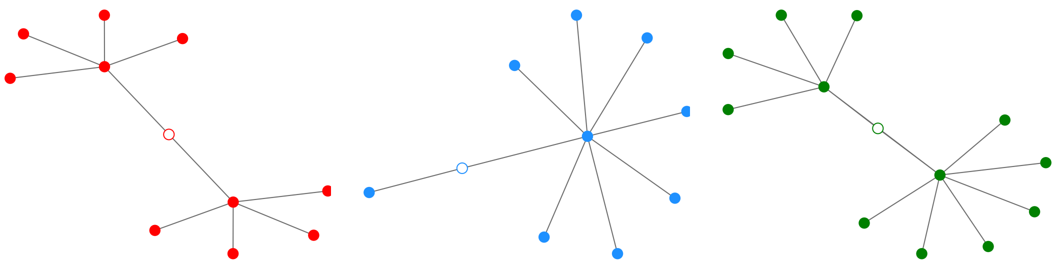
This approach allows an optimization over graph states with up to 15 qubits. In practice, the dominant cost in evaluating $I_c(\mathcal{N}_{\mathbf{p}}^{\otimes k_A})$ for a given graph is the computation of the eigenvalue distributions λ_B and λ_{BR}. Although the graph-state structure and the algorithm of [5, Algorithm 1] avoid explicit diagonalization of a $2^k \times 2^k$ density matrix, the number of amplitudes that must be tracked still grows exponentially in the total number of vertices k. This exponential scaling is the main bottleneck in our approach and is what limits our numerical study to graphs with up to 15 qubits.

4 Results

We perform extensive numerical optimization using a genetic algorithm to discover new families of graph states, beyond those of [5], that have a higher threshold than repetition codes for Pauli channels. The "threshold" for these one-parameter families of channels refers to the value of the noise parameter up to which the coherent information at a specific code remains positive. Our findings include, for the depolarizing channel, a graph state on 11 vertices exhibiting a threshold higher than the 5-qubit repetition code. Additionally, we present novel families of graph-state codes for the BB84 channel and the 2-Pauli channel achieving higher thresholds than repetition codes.

For the depolarizing and BB84 channels, the thresholds of the graph states in Figures 1a and 1c exceed both the corresponding repetition-code thresholds and the hashing bound. The graph state in Figure 1a also happens to have a higher threshold than the 5-repetition code for the BB84 channel.



(a) Depolarizing Channel (b) 2-Pauli Channel (c) BB84 Channel

Figure 1: Examples of graph states found via genetic algorithm optimization that yield positive coherent information beyond the threshold for the 5-qubit repetition code: (a) An 11-vertex graph for the depolarizing channel, (b) a 10-vertex graph for the 2-Pauli channel, and (c) a 13-vertex graph for the BB84 channel. The system vertices are filled with solid color, the only reference vertex is filled white.

Note that for the 2-Pauli channel, while the graph state in Figure 1b has a higher threshold than known repetition codes and even the 5-in-5 concatenated code, the threshold is less than the hashing bound of $p \sim 0.227092$ for the 2-Pauli channel. For the depolarizing and BB84 channels, the numerical gains in error thresholds beyond the hashing bound for the best graphs found by the genetic algorithm are on the order of 10^{-3}, which is typical for known code constructions that demonstrate superadditivity of coherent information for Pauli channels.

Channel	Baseline code [5]	p_{baseline}	k_A	p_{ours}	Δp
Depolarizing	1-in-5	0.19035609	10	0.19042509	$+6.9002388 \times 10^{-5}$
2-Pauli	5-in-5	0.22683627	9	0.22689416	$+5.7891074 \times 10^{-5}$
BB84	1-in-5	0.11210421	12	0.11212831	$+2.4096473 \times 10^{-5}$

Table 1: Placing our graph-state thresholds in the context of codes from [5]. Here $\Delta p = p_{\text{ours}} - p_{\text{baseline}}$ is the gain in the threshold p_{ours} of graphs found by our approach beyond the baseline code thresholds p_{baseline}. For reference, the 2-level tree with 13 vertices T_{13} in [5] has $p_{\text{BB84}}(\text{T}_{13}) = 0.11201907$ and $p_{\text{dep}}(\text{T}_{13}) = 0.19040260$. For the 2-Pauli channel, our approach finds a graph with 9 channel vertices that has a higher threshold than even the 5-in-5 code, which has a larger blocklength of 25.

A curious feature of the graphs in Fig. 1a and 1c is that they have large automorphism groups. The automorphism groups of the three graphs in Figs. 1a, 1b, 1c are $S_4 \wr S_2$, S_7 and $S_4 \times S_6$ respectively, where S_n is the symmetric group and $S_4 \wr S_2$ is a wreath product of groups. Such symmetry groups persist in the best graphs found by our optimization as the number of vertices increases. For example, when optimizing the coherent information of graph states with 13 vertices with depolarizing noise, we find a graph with automorphism group $S_4 \times S_6$ that also turns out to surpass the 5-repetition code threshold.

A key difference between the graphs we find and the 2-level trees considered in [5] is that the reference vertex in our graphs is not a leaf. This allows for a larger automorphism group for a fixed number of vertices. These properties make this new class of graphs found by the genetic algorithm particularly amenable to the symmetry-based framework of [5].

5 Conclusion

We cast the search for high-threshold quantum codes as an optimization over graph states and demonstrated that a basic genetic algorithm is sufficient to outperform repetition codes on several Pauli channels. On the depolarizing channel we identify an 11-vertex graph whose threshold exceeds the 5-qubit repetition code, and on BB84 and 2-Pauli we likewise improve on repetition code thresholds, while remaining below the hashing bound for the 2-Pauli channel. The best-performing graphs exhibit large and structured automorphism groups, and this pattern persists as the number of vertices grows, which points to symmetry as a practical inductive bias for constructive code families. Our results enlarge the set of graph-structured codes with positive coherent information at higher noise levels than repetition codes, and they do so with a transparent search procedure. The approach is simple to reproduce and extend. A natural direction for future investigation is the use of more efficient symmetry-based algorithms for the evaluation of the coherent information, which is the main bottleneck for our approach.

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